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POLES OF A TWO-VARIABLE P-ADIC COMPLEX POWER

BY

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ABSTRACT. For almost all P-adic completions of an algebraic number field, if $s \in \mathbf{C}$ is a pole of $f^s = \int\int |f(x, y)|^s |dx|_{K_p} |dy|_{K_p}$, where f is a polynomial whose only singular point is the origin, $f(0,0) = 0$, and f is irreducible in $\bar{K}[[x, y]]$, then $\mathrm{Re}(s)$ is -1 or one of an explicitly given set of rational numbers, whose cardinality is the number of characteristic exponents of $f = 0$.

0. Introduction. Let K be an algebraic number field, K_p a P-adic completion of K with ring of integers R, maximal ideal P, group of units $R^{\times}$, and residue class field R/P of cardinality q. The Haar measure on K_p such that R has measure 1 is called the usual Haar measure, and its product measure is the usual Haar measure on K_p^n. The absolute value $|\ |_{K_p}$ on K_p is defined as

$$|0|_{K_p} = 0 \quad \text{and} \quad |d(tx)|_{K_p} = |t|_{K_p} |dx|_{K_p}$$

for every t in $K_p - \{0\}$.

Let $f \in K[x, y]$ have a singularity only at $(0,0)$, $f(0,0) = 0$, and suppose that f is irreducible in $\bar{K}[[x, y]]$, where $\bar{K}$ is the algebraic closure of K.

Our purpose is to investigate the poles of the meromorphic continuation of the complex-valued function

$$f^s = \int_P\int_P |f(x, y)|_{K_p}^s |dx|_{K_p} |dy|_{K_p},$$

where s is a complex variable.

Igusa has given [**4**, p. 310], in a more general setting, a set of candidates which contains the poles of f^s and, in the situation described above, has determined the pole of f^s closest to the origin [**3**, p. 367].

Here we show that for almost all P-adic completions of K, if s is a pole of f^s, $\mathrm{Re}(s)$ is -1 or one of an explicitly given set of quotients of integers called "numerical data" of desingularization. Only one such quotient is associated with each characteristic exponent of $f = 0$.

Every exceptional curve in the desingularization of $f = 0$, not only the relatively few we associate below with the characteristic exponents, has a pair of numerical data whose quotient appears, at first glance, to give a negative real pole of f^s. We eliminate false candidates by using a relationship between the numerical data, and also an argument involving the Newton polygon. In the process, the behavior of a previously studied function defined on the set of exceptional curves is clarified.

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The analogous f^s for **R** has been studied [**1**–**5**, and **8**].

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1. Numerical data associated with exceptional curves. In this section we review results of Igusa [**3**]. Characteristic exponents are defined, a desingularization is described, and certain "numerical data" associated with this desingularization are given explicitly.

Let $\mathfrak{o}$ denote the local ring of an irreducible plane algebroid curve $f = 0$ over an algebraically closed field K of characteristic 0 and $\mathfrak{m}$ the maximal ideal of $\mathfrak{o}$; then we have

$$\mathfrak{m} = \mathfrak{o}x + \mathfrak{o}y$$

for some x, y in $\mathfrak{m}$. Let "ord" denote the normalized discrete valuation on the field of quotients of $\mathfrak{o}$; then the integral closure of $\mathfrak{o}$ is the ring of formal power series in any element of order 1 with coefficients in K. We shall assume that $\mathfrak{o}$ is not regular, i.e., that $\mathrm{ord}(x)$, $\mathrm{ord}(y) \geqslant 2$.

If $\mathrm{ord}(x) = m$, then $x^{1/m}$ is an element of the field of quotients of $\mathfrak{o}$ and is of order 1. We have

$$y = y(x) = \sum_{i=1}^{\infty} a_i x^{i/m}$$

with a_i in K for $i = 1, 2, \ldots$. We rewrite this "Puiseux series" as

$$y(x) = \sum_{i=1}^{k_0} a_{0,i} x^i + \sum_{i=0}^{k_1} a_{1,i} x^{(\mu_1 + i)/\nu_1} + \cdots + \sum_{i=0}^{\infty} a_{g,i} x^{(\mu_g + i)/\nu_1\nu_2\cdots\nu_g},$$

in which the exponents are strictly increasing, $a_{1,0}a_{2,0} \cdots a_{g,0} \neq 0$, μ_i, ν_i are relatively prime integers for $1 \leqslant i \leqslant g$, and $\nu_1, \nu_2, \ldots, \nu_g \geqslant 2$. We then have

$$\mathrm{ord}(x) = m = \nu_1\nu_2 \cdots \nu_g,$$

and the g exponents μ_1/ν_1, $\mu_2/\nu_1\nu_2, \ldots, \mu_g/\nu_1\nu_2 \cdots \nu_g$ are called characteristic exponents of the series $y(x)$.

Now let X denote a nonsingular algebraic surface over an algebraically closed field K (of characteristic 0) and C an irreducible curve on X which is analytically irreducible at its only singular point. It is well known that C can be desingularized through a unique series of quadratic transformations which can be described by the characteristic exponents of the corresponding algebroid curve; i.e. the total transform C^* of C under the product morphism $X^* \to X$ is desingularized; cf. [**10**, pp. 5–10]. Igusa has formulated [**3**] a quantitative theorem concerning this process, which we present after recalling some of the details of the desingularization.

Let μ_1/ν_1, $\mu_2/\nu_1\nu_2, \ldots, \mu_g/\nu_1\nu_2 \cdots \nu_g$ denote the characteristic exponents [**3**, p. 358] of C, and expand each $\mu_i/\nu_i - \mu_{i-1}$, $1 \leqslant i \leqslant g$, where $\mu_0 = 0$, into a continued fraction $\mu_i/\nu_i - \mu_{i-1} = [k_{i0}, k_{i1}, \ldots, k_{i,t_i}]$; the k_{ij} are nonnegative integers and k_{10}, $k_{i1}, \ldots, k_{i,t_{i-1}} \geqslant 1$, $k_{i,t_i} \geqslant 2$, $t_i \geqslant 1$ for $1 \leqslant i \leqslant g$. We note that unlike $k_{10} \geqslant 1$ we may have $k_{i0} = 0$ for some i. The number of quadratic transformations is the sum of all

k_{ij}. If we let C' denote the strict transform of C under the morphism $X^* \to X$, and if E_I denotes the exceptional curve of the Ith quadratic transformation, then the total transform C^* of C is of the form $C^* = \Sigma_I N_I E_I + C'$, where $N_I \geqslant 1$ for every I. By making X smaller if necessary, we may assume there exists a gauge-form $\tilde{w}$ on X, i.e., a 2-form on X without zeros or poles. Let $\tilde{w}^*$ denote the preimage of $\tilde{w}$ under $X^* \to X$; then its divisor $(\tilde{w}^*)$ is of the form $(\tilde{w}^*) = \Sigma_I (n_I - 1)E_I$, in which $n_I \geqslant 2$ for every I, and it is independent of the choice of $\tilde{w}$. We will call (N_I, n_I) the numerical data associated with E_I.

In order to study (N_I, n_I), Igusa introduces polynomials p, a, b, c, P as follows: Let $p_n = p_n(k_0, k_1, \ldots, k_n)$ denote a polynomial in $n + 1$ variables with integer coefficients defined inductively as follows: it represents 0, 1, respectively, for $n = -2$, -1 and $p_n = k_0 p_{n-1}(k_1, \ldots, k_n) + p_{n-2}(k_2, \ldots, k_n)$ for $n \geqslant 0$. Since there will be no confusion, we shall drop n from p_n. In the following we shall fix a positive integer t and limit ourselves to the $t + 1$ variables $k_0, k_1, \ldots, k_t$. For any integer pair (r, s) satisfying $0 \leqslant r \leqslant s \leqslant t$ we put

$$a(r, s) = \sum_{i=s-r}^{s} k_i p(k_{i+1}, \ldots, k_s) p(k_{i+1}, \ldots, k_t);$$

then, if we put $a(s, s) = a(s)$, we get

$$a(s) = \begin{cases} p(k_0, \ldots, k_s) p(k_1, \ldots, k_t), & s \text{ even}, \\ p(k_1, \ldots, k_s) p(k_0, \ldots, k_t), & s \text{ odd}; \end{cases}$$

this remains valid for $s = -2, -1$ if we put $a(-2) = a(-1) = 0$. We define $b(s)$ for $-2 \leqslant s \leqslant t$ as

$$b(s) = a(s) + p(k_{s+2}, \ldots, k_t);$$

then $b(t) = a(t)$. Finally, we define $c(s)$ for $s \geqslant 0$ as

$$c(s) = \sum_{i=0}^{s} k_i p(k_{i+1}, \ldots, k_s) + 1,$$

and we put $c(-2) = c(-1) = 1$; then we get $c(s) = p(k_0, \ldots, k_s) + p(k_1, \ldots, k_s)$ for $s \geqslant -1$. For $0 \leqslant s \leqslant t$ we have

$$a(s) = k_s b(s - 1) + a(s - 2), \quad b(s) = k_s a(s - 1) + b(s - 2),$$
$$c(s) = k_s c(s - 1) + c(s - 2).$$

We define

$$P(s) = \begin{cases} c(s) - a(s)/b(-1), & s \text{ even}, \\ c(s) - b(s)/b(-1), & s \text{ odd}, \end{cases}$$

for $-1 \leqslant s \leqslant t$, and find $P(-1) = 0$, $P(0) = 1$, $P(-2) = 1$, and

$$P(s) = k_s P(s - 1) + P(s - 2) \quad \text{for } 0 \leqslant s \leqslant t.$$

We may assume that X is an affine plane minus a finite number of points, C is an irreducible curve on X with the origin 0 of X as its only singular point, and C is analytically irreducible at 0; then C is defined by $f = 0$, and X has a gauge form $\tilde{w}$.

We shall denote by X_I the Ith quadratic transform of X and by 0_I the unique point of X_I where the strict transform of C and the Ith exceptional curve intersect. We denote by $f_I = 0$ and $\tilde{w}_I$ a local equation for the strict transform of C and a local gauge-form, respectively, both on X_I around 0_I.

Choosing affine coordinates (x, y) on X such that $x = 0$ is not tangent to C at 0, we get a Puiseux series

$$y = y(x) = \sum_{i=1}^{k_0} a_{0,i}x^i + \sum_{i=0}^{k_1} a_{1,i}x^{(\mu_1+i)/\nu_1} + \cdots$$

such that its characteristic exponents $\mu_1/\nu_1,\ \mu_2/\nu_1\nu_2, \ldots, \mu_g/\nu_1\nu_2\cdots\nu_g$ are also those of the algebroid curve $f = 0$ at 0. If we denote by $\varepsilon(x, y)$ a unit in the ring of formal power series in x, y, then we can write

$$f(x, y) = \varepsilon(x, y)\prod(y - \operatorname{conj} y(x)),$$

in which the product extends over $\nu_1\nu_2\cdots\nu_g$ conjugates of $y(x)$; we shall use $dx \wedge dy$ as $\tilde{w}(x, y)$. As before, we put $\mu_i/\nu_i - \mu_{i-1} = [k_{i0}, k_{i1}, \ldots, k_{i,t_i}]$ and $I_i = k_{10} + k_{11} + \cdots + k_{i,t_i}$ for $1 \leqslant i \leqslant g$, in which $\mu_0 = 0$; we denote by $a_i(s)$, $b_i(s)$, $c_i(s)$, $P_i(s)$ the $a(s)$, $b(s)$, $c(s)$, $P(s)$ for the sequence $k_{i0}, k_{i1}, \ldots, k_{i,t_i}$. Then we have the following lemma:

LEMMA 1. *If we take $I = k_{10} + k_{11} + \cdots + k_{1,s-1} + j$, where $1 \leqslant j \leqslant k_{1,s}$, $0 \leqslant s \leqslant t_1$, then we get*

$$N_I = (a_1(s-2) + jb_1(s-1))\nu_2\cdots\nu_g, \quad n_I = c_1(s-2) + jc_1(s-1).$$

The proof goes as follows: Local coordinates (u, v) are constructed on X_I valid in an open subset containing 0_I. For the sake of simplicity, omit "1" from k_{10}, $k_{11}, \ldots, k_{1,t_1}$, $a_1(s)$, $b_1(s)$, $c_1(s)$, $P_1(s)$ and put $q_i = p(k_{i+1}, \ldots, k_t)$ for $-1 \leqslant i \leqslant t+1$. By passing from (x, y) to (x', y'), defined as $x' = x$, $y' = y - \sum_{i=1}^{k_0} a_{0,i}x^i$, we may assume that $a_{0,1} = \cdots = a_{0,k_0} = 0$. Put $x = y_{-1}$, $y = x_{-1}$ and introduce $(x_0, y_0), \ldots, (x_{s-1}, y_{s-1})$ as $x_{i-1} = x_i^{k_i}y_i$, $y_{i-1} = x_i$ for $0 \leqslant i < s$; finally put $x_{s-1} = u^jv$, $y_{s-1} = u$. Repeatedly applying an inversion formula [**3**, p. 359], we get

$$f(x, y) = \left(u^{a(s-2)+jb(s-1)}v^{a(s-1)}\right)^{\nu_2\cdots\nu_g} f_I(u, v),$$
$$\tilde{w}(x, y) = \pm u^{c(s-2)+jc(s-1)-1}v^{c(s-1)-1}\, du \wedge dv.$$

We then have a Puiseux series

$$x_{s-1} = x_{s-1}(y_{s-1}) = (a_{1,0})^{(-1)^s q_0/q_s} y_{s-1}^{q_{s-1}/q_s} + \cdots,$$

and this series has

$$\frac{\mu_i - (k_0q_0 + \cdots + k_{s-1}q_{s-1} - q_0 + q_s)\nu_2\cdots\nu_i}{q_s\nu_2\cdots\nu_i}$$

for $1 \leqslant i \leqslant g$ as its characteristic exponents, unless $s = t$, in which case the above g exponents become $\mu_i/\nu_2\cdots\nu_i - \mu_1 + k_t$ for $1 \leqslant i \leqslant g$, and we simply omit the first exponent k_t which is an integer. When $I = I_1$, we get

$$v = v(u) = \sum_{i=0}^{\kappa_0} \alpha_{0,i}u^i + \alpha_{1,0}u^{\mu_2/\nu_2-\mu_1} + \cdots$$

in which $\kappa_0 = k_{2,0}$ and $\alpha_{0,0} = (a_{1,0})^{\pm\nu_1} \neq 0$, $\alpha_{1,0} \neq 0, \ldots$. Passing from (u, v) to (ξ, η) defined as $\xi = u$, $\eta = v - \sum_{i=0}^{\kappa_0} \alpha_{0,i} u^i$, we get

$$f(x, y) = \xi^{\mu_1\nu_1\cdots\nu_g} f_{I_1}(\xi, \eta), \quad \tilde{w}(x, y) = \xi^{\mu_1+\nu_1-1}\tilde{w}_{I_1}(\xi, \eta),$$

and the Puiseux series $\eta = \eta(\xi)$ has $\mu_i/\nu_2 \cdots \nu_i - \mu_1$ for $1 < i \leqslant g$ as its characteristic exponents. Lemma 1 is applied to $f_{I_1}(\xi, \eta)$ to determine N_I, n_I for $I_1 < I \leqslant I_2$, and in this way we obtain

LEMMA 2. *For $1 < i \leqslant g$ we take $I = I_{i-1} + k_{i0} + k_{i1} + \cdots + k_{i,s-1} + j$ in which $1 \leqslant j \leqslant k_{i,s}$, $0 \leqslant s \leqslant t_i$; then we get*

$$N_I = \big(P_i(s-2) + jP_i(s-1)\big)N_{I_{i-1}} + \big(a_i(s-2) + jb_i(s-1)\big)\nu_{i+1}\cdots\nu_g,$$

$$n_I = \big(P_i(s-2) + jP_i(s-1)\big)\big(n_{I_{i-1}} - 1\big) + c_i(s-2) + jc_i(s-1).$$

Moreover, if we put $M_i = N_{I_i}/\nu_{i+1}\cdots\nu_g$, $m_i = n_{I_i}$, for $1 \leqslant i \leqslant g$, we get $M_1 = \mu_1\nu_1$, $m_1 = \mu_1 + \nu_1$; and

$$M_i = (M_{i-1} + \mu_i/\nu_i - \mu_{i-1})\nu_i^2, \quad m_i = (m_{i-1} + \mu_i/\nu_i - \mu_{i-1})\nu_i.$$

Further consideration of polynomials p, a, b, c, P enables Igusa to prove his

THEOREM 1. *We put $I_i = k_{10} + k_{11} + \cdots + k_{i,t_i}$ for $1 \leqslant i \leqslant g$; then we have*

$$n_{I_1}/N_{I_1} = (1 + \nu_1/\mu_1)/\nu_1\nu_2\cdots\nu_g,$$

$$n_I/N_I > n_{I_i}/N_{I_i} \quad (I < I_1), \qquad n_I/N_I > n_{I_i}/N_{I_i} \quad (I > I_i)$$

for $1 \leqslant i < g$.

The function $I \to n_I/N_I$ is strictly decreasing in the subinterval $k_{10} + \cdots + k_{1,s-1} < I \leqslant k_{10} + \cdots + k_{1,s}$ for $0 \leqslant s \leqslant t_1$. In every other interval $I_{i-1} < I \leqslant I_i$, it is oscillating, i.e., it is strictly increasing or decreasing in the subinterval $I_{i-1} + k_{i0} + \cdots + k_{i,s-1} < I \leqslant I_{i-1} + k_{i0} + \cdots + k_{i,s}$ according as s is even or odd for $0 \leqslant s \leqslant t_i$, $1 < i \leqslant g$.

For our later purpose we add the following remark: suppose that K_0 is a subfield of K over which X, C are defined and the singular point is rational. Then all successive quadratic transformations are defined over K_0.

We have finished our review of material in [**3**].

2. Numerical data—a relationship between them, and the ordering of their quotients. Our demonstration that certain candidates fail to be poles of f^s will depend on the following relationship between the numerical data of a given exceptional curve and of those other exceptional curves it intersects.

THEOREM 1. *Suppose $I \neq I_i$, $1 \leqslant i \leqslant g$, and that, in C^*, E_I intersects exceptional curves we shall call $E_{I,3}$ (which intersects E_I at $y_I^{-1} = 0$) and $E_{I,2}$. Here, if E_I intersects only one other exceptional curve $E_{I,2}$, we assign numerical data $(0, 1)$ to a fictitious $E_{I,3}$. Then*

$$\frac{n_{I,2} + n_{I,3}}{n_I} = \frac{N_{I,2} + N_{I,3}}{N_I} = \begin{cases} 2 & \text{if } j < k, \\ k_{s+1} + 1 & \text{if } j = k_s,\ s = t-1, \\ k_{s+1} + 2 & \text{if } j = k_s,\ s < t-1. \end{cases}$$

PROOF. Considering each possible location of E_I in C^*, we form the above ratios from Igusa's expressions for the numerical data, given here in §1, and verify the equalities through the properties, also given here in §1, of the polynomials p, a, b, c, P. In the following table E_{I_0} denotes a fictitious exceptional curve with numerical data (0, 1), and s_0 is any fixed value of s.

Case #	E_I	$E_{I,3}$	$E_{I,2}$
1	$j > 1, j < k$	$j-1$	$j+1$
2	$j=1, s=0, k_0=1, t=1$	$E_{I_{i-1}}$	$j=k_1, s=t=1$
3	$j=1, s=0, k_0=1, t>1$	$E_{I_{i-1}}$	$j=1, s=2$
4	$j=1, s=0, k_0>1$	$E_{I_{i-1}}$	$j=2, s=0$
5	$j=1, s=2, k_0=0, k_2=1, t=3$	$E_{I_{i-1}}$	$s=3, j=k_3$
6	$j=1, s=2, k_0=0, k_2=1, t>3$	$E_{I_{i-1}}$	$j=1, s=4$
7	$j=1, s=2, k_0=0, k_2>1$	$E_{I_{i-1}}$	$j=2, s=2$
8	$j=1, s=1, k_1=1, t=2$	E_{I_0}	$j=k_2, s=2$
9	$j=1, s=1, k_1=1, t>2$	E_{I_0}	$j=1, s=3$
10	$j=1, s=1, k_1>1$	E_{I_0}	$j=2, s=1$
11	$j=1, 1<s_0<t-1, k_{s_0}=1$	$j=k_{s_0-2}, s=s_0-2$	$j=1, s=s_0+2$
12	$j=k_{t-1}, s=t-1>1, k_{t-1}=1$	$j=k_{t-3}, s=t-3$	$j=k_t, s=t$
13	$j=1, 1<s_0<t, k_{s_0}>1$	$j=k_{s_0-2}, s=s_0-2$	$j=2, s=s_0$
14	$j=k_{s_0}>1, s_0<t-1$	$j=k_{s_0}-1, s=s_0$	$j=1, s=s_0+2$
15	$j=k_{t-1}>1, s=t-1$	$j=k_{t-1}-1, s=t-1$	$j=k_t, s=t$

We give details in two representative cases:

Case 9. E_I has $j=1$, $s=1$. $E_{I,3}$ has $N_{I,3}=0$, $n_{I,3}=1$. $E_{I,2}$ has $j=1$, $s=3$. $k_1=1$; $t>2$.

$$\frac{n_{I,2}+n_{I,3}}{n_I}=\frac{(P(1)+P(2))(n_{I_{i-1}}-1)+c(1)+c(2)+1}{(P(-1)+P(0))(n_{I_{i-1}}-1)+c(-1)+c(0)}.$$

$P(-1)=0$; $P(0)=1$; $P(1)=k_1=1$;
$P(2)=p(k_1,k_2)=k_1k_2+1=k_2+1$.
$(P(1)+P(2))/(P(-1)+P(0))=k_2+2$.
$c(1)=k_1c(0)+c(-1)=c(0)+1$, which is k_0+2,
because $c(0)=p(k_0)+p(k_1,k_0)=k_0+1$.
$c(-1)=1$.
$c(2)=p(k_0,\ldots,k_2)+p(k_1,k_2)=k_0k_2+k_0+2k_2+1$.
$c(1)+c(2)+1=(k_0+2)(k_2+2)$.
$c(-1)+c(0)=k_0+2$.
$(c(1)+c(2)+1)/(c(-1)+c(0))=k_2+2$.

$$\frac{N_{I,2}+N_{I,3}}{N_I}=\frac{(P(1)+P(2))N_{I_{i-1}}+(a(1)+b(2))\nu_{i+1}\cdots\nu_g}{(P(-1)+P(0))N_{I_{i-1}}+(a(-1)+b(0))\nu_{i+1}\cdots\nu_g}.$$

$$\frac{P(1)+P(2)}{P(-1)+P(0)}=\frac{a(1)+b(2)}{a(-1)+b(0)}=k_2+2.$$

Case 12.

$$\frac{P(t-5)+k_{t-3}P(t-4)+P(t-2)+k_tP(t-1)}{P(t-3)+P(t-2)}$$
$$=\frac{c(t-5)+k_{t-3}c(t-4)+c(t-2)+k_tc(t-1)}{c(t-3)+c(t-2)}$$
$$=\frac{a(t-5)+k_{t-3}b(t-4)+a(t-2)+k_tb(t-1)}{a(t-3)+k_{t-1}b(t-2)}=k_t+1.$$

The numerator of the last fraction is:

$$\begin{aligned}a(t-3)+a(t-2)+k_tb(t-1)&=a(t)+a(t-3)=b(t)+a(t-3)\\&=k_ta(t-1)+b(t-2)+a(t-3)\\&=k_ta(t-1)+a(t-1)=(k_t+1)a(t-1).\end{aligned}$$

The denominator of that fraction,

$$a(t-3)+k_{t-1}b(t-2)=a(t-1).$$

Theorem 1 and its usefulness in the evaluation of f^s were introduced in the author's doctoral thesis [7].

COROLLARY 1. (1) *The function* $I\to n_I/N_I$ *is strictly decreasing as* I *ranges through the sequence of subintervals*: $0<I\leqslant k_{10}$, $k_{10}+k_{11}<I\leqslant k_{10}+\cdots+k_{12},\ldots,k_{10}+\cdots+(k_{1,t_1-1}$ *or* k_{1,t_1} *as* t_1 *is odd or even*), *and also as* I *ranges through the sequence of subintervals*: $k_{10}<I\leqslant k_{10}+k_{11}$, $k_{10}+\cdots+k_{12}<I\leqslant k_{10}+\cdots+k_{13},\ldots,k_{10}+\cdots+(k_{1,t_1-1}$ *or* k_{1,t_1} *as* t_1 *is even or odd*), *with* n_{I_1}/N_{I_1} *a lower bound for the function on both sequences, attained at* $I=k_{10}+\cdots+k_{1,t_1}$.

(2) *For* $i>1$ *the function* $I\to n_I/N_I$ *is strictly increasing as* I *ranges through the sequence of subintervals*: $I_{i-1}<I\leqslant I_{i-1}+k_{i0}$, $I_{i-1}+\cdots+k_{i1}<I\leqslant I_{i-1}+\cdots+k_{i2},\ldots,I_{i-1}+\cdots+(k_{i,t_i-1}$ *or* k_{i,t_i} *as* t_i *is odd or even*), *and strictly decreasing as* I *ranges through the sequence of subintervals*: $I_{i-1}+k_{i0}<I\leqslant I_{i-1}+\cdots+k_{i1}$, $I_{i-1}+\cdots+k_{i2}<I\leqslant I_{i-1}+\cdots+k_{i3},\ldots,I_{i-1}+\cdots+(k_{i,t_i-1}$ *or* k_{i,t_i} *as* t_i *is even or odd*), *with* n_{I_i}/N_{I_i} *an upper bound for the former and a lower bound for the latter, attained at* $I=I_i$.

(3) $n_I/N_I>n_{I_i}/N_{I_i}$ *for* $I>I_i$, $1\leqslant i<g$.

PROOF. Plotting (N_I,n_I) on a cartesian coordinate plane, our Theorem 1 tells us to obtain (N_I,n_I) for $I\neq I_i$, $1\leqslant i\leqslant g$, by adding vectors $(N_{I,2},n_{I,2})$ and $(N_{I,3},n_{I,3})$ and dividing by the scalar 2, $k_{s+1}+1$, or $k_{s+1}+2$. The slope of the first vector, which is n_I/N_I, is intermediate between the slopes of the latter two. This fact together with Igusa's Theorem 1, given here in §1, gives the corollary.

COROLLARY 2. E_{I_i}, *where* $1<i\leqslant g$, *intersects exceptional curves with quotients* n_I/N_I *both less than and greater than its own.*

3. The Newton polygon. Let K be an algebraically closed field of characteristic 0, $f(x,y)=\sum_{i=1}^n a_ix^{\alpha_i}y^{\beta_i}$ a polynomial with $a_i\in K$, such that $f(0,0)=0$, and $(0,0)$ is the only singular point of f. The Newton diagram of $f(x,y)$ is formed by plotting in a cartesian coordinate system the points P_i with coordinates $u = \alpha_i$, $v = \beta_i$. Suppose that points P_j, P_k of the Newton diagram of f are such that there exist ν_1, δ_1 in $\mathbf{R}$ satisfying $\alpha_j + \nu_1\beta_j = \alpha_k + \nu_1\beta_k = \delta_1 \leqslant \alpha_i + \nu_1\beta_i$ for $i = 1,\ldots,n$ with $\alpha_j < \alpha_k$, $\nu_1 > 0$. We define the Newton polygon of f to be the longest convex polygonal arc, each of whose vertices is a P_i, such that no P_i lies below the arc. Then P_j, P_k are on a segment L, of negative slope, of the Newton polygon. We will construct, following an argument of Walker **[9]**,

$$\bar{y} = c_1x^{\nu_1} + c_2x^{\nu_1+\nu_2} + c_3x^{\nu_1+\nu_2+\nu_3} + \cdots \quad \text{in } K(x)^*,$$

the fractional power series in x, such that $f(x, \bar{y}) = 0$. Here $c_i \neq 0$ and $\nu_2 > 0$, $\nu_3 > 0, \ldots$. There may be a finite or infinite set of c_i.

Abbreviate $\bar{y} = x^{\nu_1}(c_1 + \bar{y}_1)$, where we have put $\bar{y}_1 = c_2x^{\nu_2} + \cdots$. Then

$$\begin{aligned} f(x, \bar{y}) &= \sum_{i=1}^{n} a_i x^{\alpha_i}\bar{y}^{\beta_i} = \sum_{i=1}^{n} a_i x^{\alpha_i}\left(x^{\nu_i}(c_1 + \bar{y}_1)\right)^{\beta_i} \\ &= \sum_{i=1}^{n} a_i x^{\alpha_i+\nu_1\beta_i}(c_1 + \bar{y}_1)^{\beta_i}, \end{aligned}$$

which we can rewrite as

$$f(x, \bar{y}) = \sum_{i=1}^{n} c_1^{\beta_i}a_i x^{\alpha_i+\nu_1\beta_i} + g(x, \bar{y}_1),$$

where g contains all the terms involving $\bar{y}_1$. As the order of $\bar{y}_1$ is $\nu_2 > 0$, each term of $g(x, \bar{y}_1)$ has order greater than some one of the terms $c_1^{\beta_i}a_i x^{\alpha_i+\nu_1\beta_i}$. The terms of lowest degree in $f(x, \bar{y})$ are those $c_1^{\beta_i}a_i x^{\alpha_i+\nu_1\beta_i}$ such that $\alpha_i + \nu_1\beta_i = \delta_1$. In order that $f(x, \bar{y}) = 0$, it is necessary that the terms of lowest order cancel, i.e. $\Sigma a_h c_1^{\beta_h} = 0$, the summation being over all values of h for which $\alpha_h + \nu_1\beta_h = \delta_1$. The existence of $c_1 \neq 0$ is guaranteed by the existence of at least two values of h, namely i and j, for which $\alpha_h + \nu_1\beta_h = \delta_1$, and the fact that K is algebraically closed.

We define $f_1(x, y_1) = x^{-\delta_1}f(x, x^{\nu_1}(c_1 + y_1))$ and, considering the root y_1 of $f_1(x, y_1) = 0$, continue the process, determining c_2, ν_2. We require $\nu_2 > 0$, $\nu_3 > 0, \ldots$, i.e. in each step we need a segment with negative slope in the Newton polygon of f_i. In each step the existence of c_i is guaranteed as that of c_1 was. Finally, in order that the y we construct be in $K(x)^*$, we must show that after a certain stage all the ν_i have a common denominator.

Suppose that P_j, P_k are the left- and right-hand ends of segment L of the Newton polygon of f. Then, $\alpha_j + \nu_1\beta_j = \alpha_k + \nu_1\beta_k$ implies $\nu_1 = (\alpha_k - \alpha_j)/(\beta_j - \beta_k) = p/q$, where p, q are relatively prime integers, and q must divide $(\beta_h - \beta_k)$ if P_h is on L. For every P_h on L we have $\beta_h = \beta_k + sq$, s being a nonnegative integer. Therefore $\Sigma_h a_h c_1^{\beta_h} = 0$ has the form $c_1^{\beta_k}\phi(c_1^q) = 0$, where $\phi(z)$ is a polynomial of degree $(\beta_j - \beta_k)/q$, such that $\phi(0) \neq 0$. If $c_1 \neq 0$ is an r-fold root, $r \geqslant 1$, of $\phi(z^q) = 0$, we have

$$\phi(z^q) = (z - c_1)^r\psi(z), \qquad \psi(c_1) \neq 0.$$

Then

$$\begin{aligned} f_1(x, y_1) &= x^{-\delta_1} \sum_{i=1}^{n} a_i x^{\alpha_i} \left(x^{\nu_i}(c_1 + y_1) \right)^{\beta_i} \\ &= x^{-\delta_1} \sum_{i=1}^{n} a_i x^{\alpha_i + \nu_1 \beta_i} (c_1 + y_1)^{\beta_i} \\ &= x^{-\delta_1} \sum_{h} a_h x^{\alpha_h + \nu_1 \beta_h} (c_1 + y_1)^{\beta_h} \\ &\quad + x^{-\delta_1} \sum_{d} a_d x^{\alpha_d + \nu_1 \beta_d} (c_1 + y_1)^{\beta_d}, \end{aligned}$$

where h runs over the values of i for which P_i is on L, and d runs over the remaining values of i. Since $\alpha_h + \nu_1 \beta_h = \delta_1$, the first sum is

$$\begin{aligned} \sum_h a_h (c_1 + y_1)^{\beta_h} &= (c_1 + y_1)^{\beta_k} \phi\left((c_1 + y_1)^q \right) \\ &= (c_1 + y_1)^{\beta_k} y_1^r \psi(c_1 + y_1), \end{aligned}$$

and we have

$$\begin{aligned} f_1(x, y_1) &= y_1^r (c_1 + y_1)^{\beta_k} \psi(c_1 + y_1) + x^{-\delta_1} \sum_d a_d x^{\alpha_d + \nu_1 \beta_d} (c_1 + y_1)^{\beta_d} \\ &= c_1^{\beta_k} \psi(c_1) y_1^r \\ &\quad + (\text{terms with } y_1 \text{ to a power greater than } r \text{ and no power of } x) \\ &\quad + (\text{terms with powers of } x). \end{aligned}$$

We consider two cases: (1) If there is no term with x to a positive power and y_1 to a power less than r, then $\bar{y}_1 = 0$ is a root of $f_1(x, y_1) = 0$, and $\bar{y} = c_1 x^{\nu_1}$ is a root of $f(x, y) = 0$. (2) If there is a term with x to a positive power and y_1 to a power less than r, then the Newton polygon of f_1 has a segment of negative slope, and there exist P_r, P_s, ν_2, δ_2 satisfying

$$\alpha_r + \nu_2 \beta_r = \alpha_s + \nu_2 \beta_s = \delta_2 \leqslant \alpha_i + \nu_2 \beta_i$$

for all i, and $\nu_2 > 0$. We can then determine c_2 as we determined c_1.

We have only to show that the successive ν_i have bounded denominators; i.e., that after a certain number of steps the value of q is always 1. The line segment of negative slope we choose in the Newton polygon of f_1 has vertical height at most r, which is at most the vertical height of the segment L in the Newton polygon of f. Thus we see that r cannot increase from step to step and must take on a constant value r_0 after a finite number of steps. Then

$$\phi(z^q) = e(z - c)^{r_0} = e z^{r_0} - \cdots \mp r_0 e c^{r_0 - 1} z \pm e c^{r_0}.$$

Since K is of characteristic 0 and r_0, e, and c are each different from 0, $r_0 d c^{r_0 - 1} \neq 0$, and so $q = 1$.

We have shown

PROPOSITION 1. *Each segment of negative slope in the Newton polygon of f corresponds to a root*

$$\bar{y} = c_1 x^{\nu_1} + c_2 x^{\nu_1+\nu_2} + c_3 x^{\nu_1+\nu_2+\nu_3} + \cdots$$

in $K(x)^*$ *of* $f(x, y) = 0$, *where* $-1/\nu_1$ *is that slope, and all* $\nu_i > 0$.

COROLLARY 1. *If f is irreducible in* $K[[x, y]]$, *the Newton polygon of f cannot have two distinct segments of negative slope.*

Returning to the setting of §3, we have

COROLLARY 2. *Set* $I_0 = 0$ *and* $f_0 = f$. *For* $I \neq I_i$, $0 \leqslant i \leqslant g$, $f_I(u, v) = \sum_{i,j} a_{ij} u^i v^j$ *has exactly one j, say* $j = m$, *such that* $a_{0m} \neq 0$.

PROOF. Each f_I is irreducible in $K[[x, y]]$, because f is (see for instance [**6**, p. 100]). In particular, x and y do not divide $f_{I-1}(x, y)$, so there is at least one j such that $a_{0j} \neq 0$ and at least one i such that $a_{i0} \neq 0$. Choose i and j as small as possible, say n and m. Corollary 1 of Proposition 1 guarantees that no point of the Newton diagram of f_{I-1} may lie below the line joining $(0, m)$ and $(n, 0)$. We examine the effect of a quadratic transformation on x^n, y^m, and an arbitrary term $x^a y^b$ of f_{I-1}. (We ignore the constant coefficients of the terms.) Suppose that $n \geqslant m$. Set $x = u$, $y = uv$. Then $x^n + x^a y^b + y^m$ becomes $u^m(u^{n-m} + u^{a+b-m} v^b + v^m)$. If $n > m$, we cannot have $a + b - m = 0$, because then (a, b) would lie below the above-mentioned line. If $n = m$, we have $u^m(1 + u^{a+b-m} v^b + v^m)$ and $I = I_i$ for some $i > 0$.

4. Poles of f^s. Let K be an algebraic number field, K_P a P-adic completion of K with R, P, $R^\times$, q, and $|\,|_{K_P}$ as in the Introduction. Let $f \in K[x, y]$ have a singularity only at $(0, 0)$, $f(0, 0) = 0$, and suppose that f is irreducible in $\bar{K}[[x, y]]$, where $\bar{K}$ is the algebraic closure of K. Furthermore, assume that our choice of P is such that $f \in R[x, y]$, and that certain constants discussed below are in R or $R^\times$.

We have prepared, in Theorem 1 and Corollary 2 of Proposition 1, to examine

$$f^s = \int_P \int_P |f(x, y)|_{K_P}^s \, |dx|_{K_P} \, |dy|_{K_P},$$

where s is a complex variable. We resolve the singularity of f following §1: Pass from (x, y) to (x', y') defined as $x' = x$, $y' = y - \sum_{i=1}^{k_0} a_{0,i} x^i$. We assume P has been chosen so that $a_{0,i} \in R$ for all i. Then $x, y \in P$ imply $x', y' \in P$, and the Jacobian of this transformation has absolute value 1. Therefore we may change all $a_{0,i}$ to 0 without affecting the integral f^s.

Set $I_0 = 0$, $f_0 = f$, and $(N_0, n_0) = (0, 1)$. Suppose now that $I \neq I_i$, $0 \leqslant i \leqslant g$, and that E_I intersects exceptional curves we call $E_{I,4}$ (at the origin of the x_I, y_I plane), $E_{I,3}$ (at $y^{-1} = 0$), and $E_{I,2}$ (in C^*). When $j = k_{t-1}$, $s = t - 1$, we note that $E_{I,4} = E_{I,2}$. If there is no $E_{I,4}$ and/or there is no $E_{I,3}$, we assign fictitious exceptional curve(s) numerical data $(0, 1)$. After we perform each quadratic transformation we will evaluate part of the integral f^s. The part remaining as we come to perform the Ith quadratic transformation, including the first one, will always be (omitting K_p from $|\ |_{K_p}$) of the form

$$\text{(A)}\qquad \int_P\int_P |x|^{N_{I-1}s+n_{I-1}-1}|y|^{N_{I-1,4}s+n_{I-1,4}-1}|f_{I-1}(x,y)|^s|dx||dy|.$$

Perform a quadratic transformation $x = u$, $y = uv$. Then $x, y \in P$ imply $v \in K_p$ and $u \in \{P \cap v^{-1}P\}$, and our integral becomes

$$\int_{K_p}\int_{P\cap v^{-1}P}|u|^{N_Is+n_I-1}|v|^{N_{I,4}s+n_{I,4}-1}|f_I(u,v)|^s|du||dv|.$$

We break the domain of integration into region $1 = P \times P$, region $2 = P \times R^\times$, and region $3 = v^{-1}P \times \{K_p - R\}$. Region 1 gives us an integral of the form (A) above, and we save it for the next quadratic transformation. To treat region 3 we change to coordinates at $v^{-1} = 0$ by setting $v = w^{-1}$, $u = wz$. Then $v \in \{K_p - R\}$, $u \in v^{-1}P$ imply $w, z \in P$, and our integral becomes

$$\int_P\int_P |z|^{N_Is+n_I-1}|w|^{N_{I,3}s+n_{I,3}-1}|g_I(z,w)|^s|dz||dw|,$$

where $g_I \in R[z,w]$. The strict transform of f intersects E_I only at $v = 0$, so g has a nonzero constant term which we assume to be a unit. Then g has constant absolute value 1 on $P \times P$ and, by the lemma,

$$\int_{P^j}|x|^s|dx|_{K_p} = (1-q^{-1})\left((1-q^{-(s+1)})^{-1} - \sum_{i=0}^{j-1}q^{-(s+1)i}\right),$$

for $j \geqslant 0$, $\mathrm{Re}(s) > -1$, the integral on region 3 is

$$(1-q^{-1})^2\left((1-q^{-(N_Is+n_I)})^{-1}-1\right)\left((1-q^{-(N_{I,3}s+n_{I,3})})^{-1}-1\right).$$

To treat region 2 we turn to Corollary 2 of Proposition 1, which tells us that for $f_I = \sum_{ij}a_{ij}u^iv^j$ there is exactly one j, say $j = m$, such that $a_{0m} \neq 0$. As $f_I(0,0) = 0$, we have $a_{00} = 0$. We assume a_{0m} is a unit, and all a_{ij} are in R. Then f_I has absolute value 1 on $R^\times \times P$ and the integral on region 2 gives

$$\int_{R^\times}\int_P|u|^{N_Is+n_I-1}|v|^{N_{I,4}s+n_{I,4}-1}|f_I(u,v)|^s|du||dv|$$
$$= (1-q^{-1})^2\left((1-q^{-(N_Is+n_I)})^{-1}-1\right).$$

Adding together the integrals for regions 2 and 3 and for the "region 3" (see Note 3 below) we get when we perform the quadratic transformation that generates $E_{I,2}$, we cover E_I and obtain

$$\left((1-q^{-(N_Is+n_I)})^{-1}-1\right)(1-q^{-1})^2\cdot G,$$

where

$$G = (1-q^{-(N_{I,2}s+n_{I,2})})^{-1} + (1-q^{-(N_{I,3}s+n_{I,3})})^{-1} - 1.$$

Note 1. $(1-q^{-a})^{-1} + (1-q^{-b})^{-1} - 1 = 0$ iff $a = -b$.

Note 2. $-N_{I,2}\cdot n_I/N_I + n_{I,2} = -(-N_{I,3}\cdot n_I/N_I + n_{I,3})$ iff $(n_{I,2}+n_{I,3})/n_I = (N_{I,2}+N_{I,3})/N_I$.

Notes 1 and 2 allow us to see that Theorem 1 implies $G = 0$ for $s = -n_I/N_I$. We have proved that the portion of our integral that covers E_I does not generate a pole at $s = -n_I/N_I$.

Note 3. In this argument we have tacitly assumed that $E_{I,3}$ is not an E_{I_i}. As we examine E_{I_i}, however, we will find that the "region 3" contribution is the same whether or not E_I is an E_{I_i}.

Suppose then that $I = I_i$ for some i; the n and m of the proof of Corollary 2 of Proposition 1 are equal, and a quadratic transformation (omitting the constant coefficients) gives

$$x_{I-1}^n + \sum_{ij} x_{I-1}^i y_{I-1}^j + y_{I-1}^n, \qquad x_{I-1} = u, y_{I-1} = uv,$$

$$u^n + \sum_{ij} u^{i+j} v^j + u^n v^n = u^n\left(1 + \sum_{ij} u^{i+j-n} v^j - v^n\right).$$

Unlike the case where $n \neq m$, there may be several terms inside the parentheses in which v appears without u; we have something of the form

$$a_0 + \sum_i a_i v^i + \sum_{jk} a_{jk} u^j v^k.$$

As before, we divide the domain of integration into region $1 = P \times P$, region $2 = P \times R^\times$, and region $3 = v^{-1}P \times \{K - R\}$. As in §1, the transformation $u' = u$, $v' = v - \Sigma_{i=0}^{\kappa} b_i u^i$, $b_0 \neq 0$ moves the origin to the intersection of E_{I_i} and the strict transform. We assume a_0, $b_0 \in R^\times$, and all a_i, b_i, $a_{jk} \in R$. Then the strict transform intersects E_{I_i} in region 2. As before, the two exceptional curves E_{I_i} intersects that have I smaller than its own we call $E_{I,2}$ and $E_{I,3}$.

We argue as before that in region 3 $|g| = 1$, and we have

$$(1 - q^{-1})^2\left((1 - q^{-(N_I s + n_I)})^{-1} - 1\right)\left((1 - q^{-(N_{I,3} s + n_{I,3})})^{-1} - 1\right).$$

Region 1 has

$$\int_P\int_P | u |^{N_I s + n_I - 1} | v |^{N_{I,2} s + n_{I,2} - 1} | \text{unit} + \text{terms in } P |^s | du || dv |$$

$$= (1 - q^{-1})^2\left((1 - q^{-(N_I s + n_I)})^{-1} - 1\right)\left((1 - q^{-(N_{I,2} s + n_{I,2})})^{-1} - 1\right).$$

Region 2 has

$$\int_{R^\times}\int_P | u |^{N_I s + n_I - 1} \left| a_0 + \sum_i a_i v^i + \sum_{jk} a_{jk} u^j v^k \right|^s | du || dv | .$$

The Jacobian of the transformation from (u, v) to (u', v') has absolute value 1, and $u \in P$, $v \in R^\times$ imply $u' \in P$, $v' \in \{P \amalg \{R^\times - \{b_0/P\}\}\}$. We divide region 2 into region $4 = P \times P$ and region $5 = P \times \{R^\times - \{b_0/P\}\}$. Region 4 is of the form (A) above, and we save it for the next quadratic transformation, unless $i = g$, in which case it covers only parts of E_{I_g} and the strict transform. Region 5 covers only part of E_{I_i}, and we do not evaluate it.

We have proved our main result:

THEOREM 2. *For almost all P-adic completions of K, the poles of*

$$f^s = \int_P \int_P |f(x, y)|^s \, |dx|_{K_p} |dy|_{K_p}$$

are on some or all of $\mathrm{Re}(s) = -1$, $-n_{I_i}/N_{I_i}$, *where* $1 \leqslant i \leqslant g$. N_I, n_{I_i} *are given explicitly in* §1 *and give possible poles of the form* $\mathrm{constant}/(1 - q^{-(N_{I_i}s + n_{I_i})})$.

We have not ruled out the possibility that $\mathrm{Re}(s) = -n_{I_i}/N_{I_i}$ may fail to give poles, except in the case $i = 1$ (see [**3**, p. 367]). Igusa's argument in that case depends on the fact that n_{I_1}/N_{I_1} is smaller than the quotients corresponding to the three components of C^* that intersect E_{I_1}; for $1 < i \leqslant g$, Corollary 3 of our Theorem 1 tells us E_{I_i} intersects exceptional curves whose quotients n_{I_i}/N_{I_i} are both larger than and smaller than its own.

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